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# Theoretical investigation of Fréedericksz transitions in twisted nematics with surface tilt

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**Abstract.** Patches of reverse tilt in a twisted nematic liquid crystal display device may be removed by inducing the same small angle of pretilt at both solid surfaces. Continuum theory is used to investigate the effect of this initial tilt upon a Fréedericksz transition in a twisted nematic in an electric field. In particular, we describe the various initial alignments that can occur and verify that the field has little influence until it approaches the threshold value corresponding to a similar cell without tilt.

## 1. Introduction

Technological interest in the commercial applications of Fréedericksz transitions in liquid crystals has grown rapidly over the last decade. Liquid crystals are anisotropic liquids composed of large, relatively rigid, rod-like molecules which tend locally to be parallel leading to transversely isotropic properties. It is common to refer to the axis of transverse isotropy as the optic axis or simply the anisotropic axis. Solid boundaries and externally applied electromagnetic fields can affect the orientation of the anisotropic axis, and a number of interesting experiments have been developed to investigate the competition between these orienting influences. In particular, one of the first Fréedericksz transitions experiments involves a sample of nematic liquid crystal at rest in a small gap between suitably prepared parallel plates in which the initial orientation of the anisotropic axis is uniformly parallel to the plane of the plates. If one applies a uniform magnetic field perpendicular to the plates there is no appreciable distortion of the initial orientation pattern until the field strength exceeds a critical value, when there is a transition to a perturbed configuration in which the anisotropic axis tilts in the direction of the field. By rotating one of the plates in its own plane relative to the other plate one can obtain a twisted configuration for the initial orientation of the anisotropic axis from the uniform parallel alignment described above. This twisted layer rotates the plane of polarisation of linearly polarised light and in the display devices proposed by Schadt and Helfrich (1971) an electric field is used to temporarily distort the twist.

The performance of these twisted nematic display devices is often spoilt by the occurrence of patches of non-uniform contrast which are in general caused by non-unique distortions of the anisotropic axis within the sample. These patches can occur in two ways, both arising from the absence of physical polarity in liquid crystals. In the first instance there can be regions with positive or negative twist, and this is readily

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remedied by incorporating cholesteric additives in the nematic liquid crystal (Raynes 1974). The second type occurs when the field is applied and the anisotropic axis in some regions may tilt in the direction of the field in a sense which is opposite to that of the rest of the sample. Raynes (1975) has shown that these patches are removed by inducing small misalignments of the anisotropic axis at the solid boundaries which are tilted in the same sense.

The occurrence of a Fréedericksz transition in devices with tilt is slightly puzzling since it is known that the electric field has an immediate effect upon the alignment between the plates. Fahrenschon *et al* (1976) discuss examples not involving twist which clearly demonstrate that there is no threshold field strength for distortion in a nematic possessing various degrees of tilt. On the other hand, Dafermos (1970) shows that slight misalignments of the magnetic field do not alter the estimate for the critical strength to cause a significant distortion in the Fréedericksz transitions experiment discussed earlier. Intuitively one would not expect small angles of pretilt to seriously affect the situation. The purpose of this paper is to clarify this question for twisted nematic liquid crystals.

We look at the case where a sample of twisted nematic liquid crystal is enclosed between parallel plates in such a way that the anisotropic axis is tilted in the same sense at both solid surfaces, the angles of tilt being small and of the same magnitude at these boundaries. Various initial solutions depending on the relative magnitudes of the Frank constants are obtained and the theory confirms the above experimental observations. Our analysis predicts an immediate change in the alignment when the field is applied, but the deviations remain small until the field strength approaches that for distortion in the corresponding untilted cell with twist. The analysis is similar to that of Dafermos (1970) when discussing small variations in the orientation of the field in the Fréedericksz transitions experiments.

## 2. Basic equations and boundary conditions

Accounts of the physical properties and associated continuum theory for liquid crystals are readily available in the book by de Gennes (1974) and in the reviews by Chandrasekhar (1976) and Stephen and Straley (1974). A detailed review of the equilibrium continuum theory is given by Ericksen (1976). Consequently, in this section, we simply summarise the basic equations.

As Ericksen (1962) discusses, continuum theory for the static isothermal behaviour of a nematic liquid crystal in an electric field $E_i$ reduces to

$$\left(\frac{\partial W}{\partial d_{i,j}}\right)_{,j}-\frac{\partial W}{\partial d_i}+\gamma d_i+(\epsilon_{\|}-\epsilon_{\perp})E_k d_k E_i \doteq 0, \tag{2.1}$$

where $W$ is the Helmholtz free energy per unit volume, $d_i(x_j)$ is a unit vector field describing the orientation of the anisotropic axis, $\gamma$ is an arbitrary constant, and $\epsilon_{\|}$ and $\epsilon_{\perp}$ are dielectric constants which are assumed to satisfy

$$\epsilon_{\|}>\epsilon_{\_}>0 \tag{2.2}$$

for this analysis. In line with similar studies, we adopt the following form for $W$, developed by Oseen (1925) and Frank (1958):

$$2W=k_2(d_{i,j})^2+k_4 d_{i,j}d_{j,i}+(k_1-k_2-k_4)(d_{i,i})^2+(k_3-k_2)d_i d_j d_{k,i} d_{k,j}, \tag{2.3}$$

where $k_1$, $k_2$, $k_3$ and $k_4$ are constants. Using energy considerations, Ericksen (1966) shows that these constants must satisfy certain inequalities. We accept these and make the further assumption that

$$k_1 > 0, \qquad k_2 > 0, \qquad k_3 > 0, \tag{2.4}$$

thereby excluding the possibility that the above coefficients vanish.

Choosing Cartesian axes so that the plates lie in the planes $z = 0$ and $z = 2l$, where $l$ is a constant, we consider solutions of equations (2.1) and (2.3) in which the orientation of the anisotropic axis takes the form

$$d_x = \cos\theta(z)\cos\phi(z), \qquad d_y = \cos\theta(z)\sin\phi(z), \qquad d_z = \sin\theta(z), \tag{2.5}$$

with

$$E_x = 0, \qquad E_y = 0, \qquad E_z = E. \tag{2.6}$$

As Deuling (1972) discusses, the field strength does not remain uniform across the gap between the plates since the field interacts with the distorted liquid crystal. However, the electric displacement $D_i$ satisfies

$$D_{i,i} = 0, \tag{2.7}$$

which implies that $D_z$ is equal to a constant value $D$. Reasoning parallel to that of Ericksen (1962) for a magnetic field results in the following constitutive equation for $D_i$:

$$D_i = \epsilon_\perp E_i + (\epsilon_\parallel - \epsilon_\perp)E_k d_k d_i. \tag{2.8}$$

Equations (2.5), (2.6), (2.7) and (2.8) then yield

$$E = \frac{D}{\epsilon_\parallel \sin^2\theta + \epsilon_\perp \cos^2\theta}. \tag{2.9}$$

By substitution of equations (2.3), (2.5), (2.6) and (2.9) in equations (2.1) and elimination of the scalar $\gamma$ it follows that

$$f(\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}(f(\theta))\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}(g(\theta))\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 + \frac{\epsilon_a D^2 \sin\theta\cos\theta}{(\epsilon_\parallel \sin^2\theta + \epsilon_\perp\cos^2\theta)^2} = 0, \tag{2.10}$$

and

$$g(\theta)\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2} + \frac{\mathrm{d}}{\mathrm{d}\theta}(g(\theta))\frac{\mathrm{d}\theta}{\mathrm{d}z}\frac{\mathrm{d}\phi}{\mathrm{d}z} = 0, \tag{2.11}$$

where

$$f(\theta) = k_1\cos^2\theta + k_3\sin^2\theta, \tag{2.12}$$

$$g(\theta) = \cos^2\theta(k_2\cos^2\theta + k_3\sin^2\theta), \tag{2.13}$$

$$\epsilon_a = \epsilon_\parallel - \epsilon_\perp. \tag{2.14}$$

Equation (2.11) integrates immediately to give

$$g(\theta)\frac{\mathrm{d}\phi}{\mathrm{d}z} = b, \tag{2.15}$$

where $b$ is an arbitrary constant. In the ensuing analysis it becomes apparent that we must investigate solutions for which both $\theta$ and $\phi$ vary. Multiplying equation (2.10) by the derivative of $\theta$ and equation (2.11) by the derivative of $\phi$, adding and integrating yields

$$f(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2+g(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2-\frac{D^2}{\epsilon_{\parallel}\sin^2\theta+\epsilon_{\perp}\cos^2\theta}=c, \tag{2.16}$$

where $c$ is an arbitrary constant. Equations (2.10) and (2.16) are similar to the corresponding equations derived by Leslie (1970) for a twisted nematic liquid crystal in a uniform magnetic field, but the terms associated with the externally applied field have a different dependence on $\theta$.

As Raynes (1975) discusses, suitable treatment of the surfaces of the parallel plates confining a small sample of nematic liquid crystal can lead to a twisted equilibrium configuration with equal angles of tilt in the same sense at both solid boundaries. In line with other static studies we assume strong anchoring of the anisotropic axis at the plates (see for example, de Gennes 1974) and impose the following boundary conditions:

$$\theta(0)=\theta(2l)=\alpha, \qquad \phi(0)=-\phi_0, \qquad \phi(2l)=\phi_0, \tag{2.17}$$

where $\alpha$ and $\phi_0$ are arbitrary constants which we may consider positive without loss of generality. Given the good agreement between theory and experiment for a number of similar transition effects, it appears reasonable to assume that the external field does not affect the orientation at the solid surfaces.

## 3. Initial solutions

In the absence of the electric field, it is natural to investigate a solution of equations (2.10) and (2.11) subject to boundary conditions (2.17) of the form

$$\theta=\alpha, \qquad \phi=\phi(z). \tag{3.1}$$

However, it turns out that such a solution is not possible unless $\alpha$ is zero, and so we seek symmetric solutions in which

$$\theta(z)=\theta(2l-z), \qquad 0\leqslant z\leqslant l, \tag{3.2}$$

$$\theta(l)=\theta_0, \qquad \left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)_{z=l}=0, \tag{3.3}$$

where $\theta_0$ is a parameter to be determined. From equation (2.15) it follows that

$$\phi(z)=-\phi(2l-z), \qquad 0\leqslant z\leqslant l, \tag{3.4}$$

and, as a result,

$$\phi(l)=0. \tag{3.5}$$

Employing equations (2.15) and (3.3) in equation (2.16) then yields

$$f(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2=b^2\left(\frac{1}{g(\theta_0)}-\frac{1}{g(\theta)}\right). \tag{3.6}$$

Thus, in view of the constraints (2.4), a necessary condition to obtain solutions of equation (3.6) is

$$0 < g(\theta_0) \leq g(\theta), \tag{3.7}$$

and so the forms of the initial solutions depend on the nature of $g(\theta)$. Leslie (1975) reaches the same conclusion in a discussion of similar solutions.

The function $g(\theta)$ is even in $\theta$ and detailed examination of its behaviour in the interval $[0, \pi/2]$ shows that it decreases monotonically to zero if

$$k_3 \leq 2k_2. \tag{3.8}$$

When $k_3$ exceeds $2k_2$, $g(\theta)$ initially increases to a maximum value at $\theta_c$, where

$$\sin^2 \theta_c = \frac{k_3 - 2k_2}{2(k_3 - k_2)}, \tag{3.9}$$

and then decreases monotonically towards zero for values of $\theta$ in the interval $(\theta_c, \pi/2)$. Therefore, as the solution of equation (3.6) depends on the relative magnitudes of the Frank constants and the magnitude of $\alpha$, it is convenient to treat separately three distinct cases.

### *3.1.* $k_3 < 2k_2$

As $g(\theta)$ decreases monotonically for $\theta \in (0, \pi/2)$ it follows from condition (3.7) that

$$\alpha \leq \theta \leq \theta_0 < \pi/2, \qquad 0 \leq z \leq l. \tag{3.10}$$

Integration of equations (2.15) and (3.6) then results in

$$bz = (g(\theta_0))^{1/2} \int_\alpha^\theta \left( \frac{f(\psi)g(\psi)}{g(\psi) - g(\theta_0)} \right)^{1/2} d\psi, \qquad 0 \leq z \leq l, \tag{3.11}$$

and

$$\phi = -\phi_0 + (g(\theta_0))^{1/2} \int_\alpha^\theta \left( \frac{f(\psi)}{g(\psi)(g(\psi) - g(\theta_0))} \right)^{1/2} d\psi, \qquad 0 \leq z \leq l. \tag{3.12}$$

The solution is then completed by equations (3.2) and (3.4), provided that the parameters $\theta_0$ and $b$ satisfy

$$bl = (g(\theta_0))^{1/2} \int_\alpha^{\theta_0} \left( \frac{f(\theta)g(\theta)}{g(\theta) - g(\theta_0)} \right)^{1/2} d\theta, \tag{3.13}$$

$$\phi_0 = (g(\theta_0))^{1/2} \int_\alpha^{\theta_0} \left( \frac{f(\theta)}{g(\theta)(g(\theta) - g(\theta_0))} \right)^{1/2} d\theta. \tag{3.14}$$

Equation (3.13) serves to evaluate the constant $b$, and equation (3.14) determines $\theta_0$ as a function of $\phi_0$.

The change of variable

$$\sin \lambda = \sin \theta / \sin \theta_0 \tag{3.15}$$

reduces equations (3.13) and (3.14) to

$$bl = (g(\theta_0))^{1/2} \int_{\lambda_0}^{\pi/2} \left( \frac{f(\theta)g(\theta)}{2k_2 - k_3 + (k_3 - k_2)(\sin^2 \theta + \sin^2 \theta_0)} \right)^{1/2} \frac{d\lambda}{\cos \theta}, \tag{3.16}$$

and

$$\phi_0 = (g(\theta_0))^{1/2} \int_{\lambda_0}^{\pi/2} \left( \frac{f(\theta)}{g(\theta)[2k_2 - k_3 + (k_3 - k_2)(\sin^2\theta + \sin^2\theta_0)]} \right)^{1/2} \frac{d\lambda}{\cos\theta}, \tag{3.17}$$

where

$$\sin\lambda_0 = \sin\alpha / \sin\theta_0. \tag{3.18}$$

Approximation of the above equations for small variations in $\alpha$ and $\theta$ gives

$$bl = \left( \frac{k_1 k_2^2}{2k_2 - k_3} \right)^{1/2} \int_{\lambda_0}^{\pi/2} d\lambda, \qquad \phi_0 = \left( \frac{k_1}{2k_2 - k_3} \right)^{1/2} \int_{\lambda_0}^{\pi/2} d\lambda, \tag{3.19}$$

and

$$\sin\lambda_0 = \alpha/\theta_0 \tag{3.20}$$

respectively. Solving these equations for $b$ and $\lambda_0$ and using equation (3.20) eventually gives

$$\theta_0 = \alpha \sec\left[ \left( \frac{2k_2 - k_3}{k_1} \right)^{1/2} \phi_0 \right]. \tag{3.21}$$

Hence $\theta_0$ does not differ significantly from $\alpha$ unless

$$\left( \frac{2k_2 - k_3}{k_1} \right)^{1/2} \phi_0 \tag{3.22}$$

is close to $\pi/2$. When the value of $\phi_0$ is $\pi/4$, as is generally the case, it follows that the maximum distortion across the plates is not too large provided $4k_1$ exceeds $(2k_2 - k_3)$ by a significant amount. Existing data on these elastic constants suggest that this condition is unlikely to be violated, except in the neighbourhood of a smectic transition (see for example, Ericksen 1976).

### 3.2. $k_3 = 2k_2$

As $g(\theta)$ decreases monotonically towards zero in $(0, \pi/2)$ $\theta$ again satisfies conditions (3.10). An inspection of the above solution for a suitable approximation when $\theta$ and $\alpha$ are both small suggests that we examine solutions for $\theta$ and $\phi$ of the form

$$\theta = \alpha + \alpha^p \Theta, \tag{3.23}$$

$$\phi = \phi_0(z - l)/l + \alpha^q \Phi, \tag{3.24}$$

where $p$ and $q$ are positive integers, and $\Theta$ and$\Phi$ are functions of $z$ such that

$$\Theta(0) = \Theta(2l) = \Phi(0) = \Phi(2l) = 0. \tag{3.25}$$

These estimates are based on the assumption that the form of the solution for small angles of tilt does not differ significantly from the solution for an untilted twisted nematic in the same configuration.

When we substitute the estimates (3.23) and (3.24) into equations (2.10) and (2.11) and retain only those terms in the lowest powers of $\alpha$, it follows that

$$p = 3, \qquad q = 6. \tag{3.26}$$

Hence first approximations to the differential equations (2.10) and (2.11) are

$$k_1 \frac{d^2\Theta}{dz^2} + \frac{2k_2\phi_0^2}{l^2} = 0, \tag{3.27}$$

$$\frac{d^2\Phi}{dz^2} - 4\frac{\phi_0}{l}\frac{d\Theta}{dz} = 0, \tag{3.28}$$

respectively. When these equations are solved subject to boundary conditions (3.25), use of equations (3.23) and (3.24) yields estimates for the tilt and twist between the plates, the maximum tilt being

$$\theta_0 = \alpha + k_2\alpha^3\phi_0^2/k_1. \tag{3.29}$$

The same estimate for $\theta_0$ also follows less directly from the integral solutions.

### *3.3* $k_3 > 2k_2$

When $\theta_c$ exists three distinct possibilities arise, depending on the relative magnitudes of $\alpha$ and $\theta_c$. If $\alpha$ is less than $\theta_c$, solutions of equation (3.6) take the form

$$0 \leqslant \theta_0 \leqslant \theta \leqslant \alpha < \theta_c, \qquad 0 \leqslant z \leqslant l. \tag{3.30}$$

We must ensure that $\theta_0$ is non-negative to avoid values of $\theta$ for which $g(\theta)$ does not satisfy condition (3.7). When $\alpha$ is greater than $\theta_c$, a necessary condition to obtain a solution of equation (3.6) is

$$\theta_c < \alpha \leqslant \theta \leqslant \theta_0 < \pi/2, \qquad 0 \leqslant z \leqslant l. \tag{3.31}$$

For the case when $\alpha$ equals $\theta_c$, both types of solution described above are equally likely. In general, since $\alpha$ is small, our interest is solely in solutions of the type (3.30). However, if $\theta_c$ is also small, we must consider all three possibilities. The discussion of solutions of type (3.31) in this event then follows from the case in § 3.1, and therefore we need only consider solutions of the type (3.30).

The form of the solution and the conditions to be satisfied by the parameters $\theta_0$ and $b$ are derived in the same way as that for § 3.1. The expressions in equations (3.11), (3.12), (3.13) and (3.14) are unchanged except for the fact that the limits are reversed in the integrals. The change of variable

$$\cosh\lambda = \sin\theta/\sin\theta_0, \tag{3.32}$$

and approximation for small variations in $\theta$ and $\alpha$ in the resulting equations for $\theta_0$ and $b$ eventually yields the following expression for the maximum distortion in $\theta$:

$$\theta_0 = \alpha\ \mathrm{sech}\left[\left(\frac{k_3 - 2k_2}{k_1}\right)^{1/2}\phi_0\right]. \tag{3.33}$$

Therefore $\theta_0$ is always smaller than $\alpha$ and so the variation in the alignment of the anisotropic axis is always small and less than the tilt at the boundaries.

## 4. Distorted solutions

In the presence of the field the initial solutions derived above are naturally no longer valid, and so we investigate solutions of equations (2.10) and (2.11) subject to boundary conditions (2.17) which are of the form in equation (3.2) with

$$\left(\frac{d\theta}{dz}\right)_{z=l}=0, \qquad \theta(l)=\theta_m, \tag{4.1}$$

where $\theta_m$ is a parameter to be determined. It may be possible to construct other solutions of a different form. However, we ignore these here and consider only solutions with the properties (3.2) and (4.1), this essentially assuming that these simple solutions are those favoured by the usual energy criterion. Using the conditions (4.1) and equation (2.15) in equation (2.16) yields

$$f(\theta)\left(\frac{d\theta}{dz}\right)^2=D^2\left(\frac{1}{\epsilon_\parallel \sin^2\theta+\epsilon_\perp \cos^2\theta}-\frac{1}{\epsilon_\parallel \sin^2\theta_m+\epsilon_\perp \cos^2\theta_m}\right)+b^2\left(\frac{1}{g(\theta_m)}-\frac{1}{g(\theta)}\right). \tag{4.2}$$

The results (3.4) and (3.5) again follow from integration of equation (2.15).

We first consider solutions for which the inclination of the anisotropic axis exceeds the tilt at the boundaries. These occur when $k_3$ is less than or equal to $2k_2$, or when $\theta_c$ is small and solutions in the interval $(\theta_c, \pi/2)$ are sought. Proceeding as before and integrating equations (2.15) and (4.2) one readily obtains

$$l=\int_\alpha^{\theta_m}\left\{f(\theta)\left[D^2\left(\frac{1}{\epsilon_\parallel \sin^2\theta+\epsilon_\perp \cos^2\theta}-\frac{1}{\epsilon_\parallel \sin^2\theta_m+\epsilon_\perp \cos^2\theta_m}\right)+b^2\left(\frac{1}{g(\theta_m)}-\frac{1}{g(\theta)}\right)\right]^{-1}\right\}^{1/2} d\theta, \tag{4.3}$$

and

$$\phi_0=\int_\alpha^{\theta_m}\left\{f(\theta)\left[D^2\left(\frac{1}{\epsilon_\parallel \sin^2\theta+\epsilon_\perp \cos^2\theta}-\frac{1}{\epsilon_\parallel \sin^2\theta_m+\epsilon_\perp \cos^2\theta_m}\right)+b^2\left(\frac{1}{g(\theta_m)}-\frac{1}{g(\theta)}\right)\right]^{-1}\right\}^{1/2} \frac{b\, d\theta}{g(\theta)}. \tag{4.4}$$

By making the change of variable

$$\sin\lambda=\sin\theta/\sin\theta_m, \tag{4.5}$$

equations (4.3) and (4.4) become

$$l=\int_{\lambda_m}^{\pi/2}\left[f(\theta)\left(\frac{\epsilon_a D^2}{(\epsilon_a \sin^2\theta+\epsilon_\perp)(\epsilon_a\sin^2\theta_m+\epsilon_\perp)}-\frac{b^2F(\theta,\theta_m)}{g(\theta)g(\theta_m)}\right)^{-1}\right]^{1/2}\frac{d\lambda}{\cos\theta}, \tag{4.6}$$

$$\phi_0=\int_{\lambda_m}^{\pi/2}\left[f(\theta)\left(\frac{\epsilon_a D^2}{(\epsilon_a \sin^2\theta+\epsilon_\perp)(\epsilon_a\sin^2\theta_m+\epsilon_\perp)}-\frac{b^2F(\theta,\theta_m)}{g(\theta)g(\theta_m)}\right)^{-1}\right\}^{1/2}\frac{b\, d\lambda}{g(\theta)\cos\theta}, \tag{4.7}$$

where

$$F(\theta,\theta_m)=k_3-2k_2-(k_3-k_2)(\sin^2\theta+\sin^2\theta_m), \tag{4.8}$$

and

$$\sin\lambda_m=\sin\alpha/\sin\theta_m. \tag{4.9}$$

Approximation of the above equations for small variations in $\alpha$ and $\theta$ and solving for $b$ and $\lambda_m$ gives

$$\theta_m = \alpha \sec\left[\left(\frac{\epsilon_a l^2 D^2/\epsilon_\perp^2 + (2k_2 - k_3)\phi_0^2}{k_1}\right)^{1/2}\right]. \tag{4.10}$$

Therefore the maximum distortion across the plates is of the same order as $\alpha$ provided

$$\left(\frac{\epsilon_a l^2 D^2/\epsilon_\perp^2 + (2k_2 - k_3)\phi_0^2}{k_1}\right)^{1/2} < \frac{\pi}{2}. \tag{4.11}$$

However, if

$$\epsilon_a l^2 E^2 \simeq k_1(\pi/2)^2 + (k_3 - 2k_2)\phi_0^2, \tag{4.12}$$

the distortion from the initial orientation pattern is significant.

When $k_3$ exceeds $2k_2$ by a finite amount, we obtain approximate initial solutions in the interval $(0, \theta_c)$ for which $\theta_0$ is less than $\alpha$. Therefore solutions in which $\theta_m$ is at first less than $\alpha$ can occur when the field is present. Such solutions are derived in a manner similar to that used above. The change of variable

$$\cosh\lambda = \sin\theta/\sin\theta_m, \tag{4.13}$$

and approximation of the resulting equations eventually yields

$$\theta_m = \alpha\ \mathrm{sech}\left[\left(\frac{(k_3 - 2k_2)\phi_0^2 - \epsilon_a l^2 D^2/\epsilon_\perp^2}{k_1}\right)^{1/2}\right]. \tag{4.14}$$

Thus the maximum distortion in this case is less than the tilt at the boundaries until

$$\epsilon_a l^2 D^2/\epsilon_\perp^2 = (k_3 - 2k_2)\phi_0^2, \tag{4.15}$$

when the tilt is uniform across the plates. For field strengths which exceed this value $\theta$ and $\theta_m$ are greater than $\alpha$, and the approximate solution follows the same lines as that discussed above. Consequently, the distortion again remain small until the field strength approaches that of equation (4.12).

In conclusion, therefore, our analysis confirms the intuitive prejudice borne out by practice that changes in the initial orientation of the anisotropic axis remain small until the electric field strength reaches the value in equation (4.12). This threshold field strength equals that intimated by Raynes (1975) for a twisted nematic without tilt in an electric field. The corresponding calculations for a magnetic field (Fraser 1976) are simpler, but are qualitatively similar to those described above, the effective field strength being equal to that for a twisted nematic without tilt (Leslie 1970). Generalising from these results and that of Dafermos (1970), we conclude that small angles of tilt do not appear to alter the threshold field strengths for significant changes in alignment in the Fréedericksz transitions experiments for nematic liquid crystals.

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